

# A new approach to Sheppard's corrections

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## Abstract

A very simple closed-form formula for Sheppard's corrections is recovered by means of the classical umbral calculus. By means of this symbolic method, a more general closed-form formula for discrete parent distributions is provided and the generalization to the multivariate case turns to be straightforward. All these new formulae are particularly suited to be implemented in any symbolic package.

**keywords:** raw moment, grouped moment, Sheppard's correction, umbral calculus, Bernoulli polynomials

## 1 Introduction

In the real world, continuous variables are observed and recorded in finite precision through a rounding or coarsening operation, i.e. a grouping rule. A compromise between the desire to know and the cost of knowing is then a necessary consequence. The literature on grouped data spans different research areas, see for example [17]. In particular, attention has been paid in the literature to the computation of moments when data are grouped into classes. Indeed, the method of moments performs estimation less well than Maximum Likelihood, but it is useful in situation when Maximum Likelihood is not feasible or has poor small sample size performance.

The moments computed by means of the resulting grouped frequency distribution are looked upon as a first approximation to the moments of the parent distribution, but they suffer from the error committed in grouping. The correction for grouping is a sum of two terms, the first depending on the length of the grouping interval, the second being a periodic function of

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the position. For a continuous random variable (r.v.), this very old problem was first discussed by Thiele [27], who studied the second term missing the first, and then by Sheppard [26] who studied the first term, missing the second. Both Bruns [4] and Fisher [15] proved that the second term can be neglected under suitable hypothesis and so for using Sheppard's corrections, that are nowadays still employed. If  $\tilde{a}_n$  denotes the  $n$ -th moment of the grouped distribution, then the  $n$ -th raw moment  $a_n$  of the continuous parent distribution can be constructed via Sheppard's corrections as

$$a_n = \sum_{j=0}^n \binom{n}{j} (2^{1-j} - 1) B_j h^j \tilde{a}_{n-j}, \quad (1)$$

where  $\{B_j\}$  are the *Bernoulli numbers*<sup>1</sup> and  $h$  is the length of the grouping intervals. The derivation of Sheppard's corrections was a popular topic in the first half of last century, see [16] for an historical account. This because Sheppard deduced equation (1) by using a suitable summation formula whose remainder term goes to zero when the density function has high order contact with the  $x$ -axis at both ends. So there was a considerable controversy on the set of sufficient conditions to be required in order to use formula (1). All these sufficient conditions can be removed, if the rounding lattice is assumed to be random and the average is made. Less research on moment corrections has appeared since the definitive work of Kendall [18], although recently the appropriateness of Sheppard's corrections was re-examined in connection with some applications, see for example [6] and [28].

Grouping includes also censoring or splitting data into categories during collection or publication, and so it does not only involve continuous variables. The derivation of corrections for raw moments of a discrete parent distribution followed a different path from Sheppard's corrections. They were first given in the Editorial of Vol.1, no. 1, of *Annals of Mathematical Statistics* (page 111). The method used to develop the general formula was extremely laborious. Some years later, Craig [5] considerably reduced and simplified the derivation of these corrections by using the logarithm of the moment generating function, that is working on cumulants instead of moments. Craig proposed the same method to derive formula (1), stating these corrections on the average and so avoiding to require any conditions on the parent distribution. At the moment, his method represents the most

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<sup>1</sup>Many characterizations of the Bernoulli numbers can be found in the literature and each could be used to define these numbers. Here we refer to the sequence of numbers such that  $B_0 = 1$  and  $\sum_{k=0}^n \binom{n+1}{k} B_k = 0$  for  $n = 0, 1, 2, \dots$

general way to find such corrections, both for continuous and for discrete parent distributions.

In this paper, we propose to overcome Craig's methods by stating Sheppard's corrections on the average through the employment of the classical umbral calculus. This approach is partially motivated by a similar employment of the classical umbral calculus in wavelets theory [23, 25]. Indeed, the reconstruction of the full moment information  $a_n$  in (1) through the grouped moments  $\tilde{a}_n$  can be seen as some kind of multilevel analysis, like in wavelets analysis. But the paper is inspired also from the belief that the classical umbral calculus can be fruitfully used in statistics, coming out the side authentically algebraic of many techniques commonly used to manage number sequences related to r.v.'s.

The umbral calculus was studied by Rota and his collaborators for long time [7]. The version introduced in [22] represents a new way of dealing with number sequences, which is represented by a symbol  $\alpha$ , called an *umbra*. More precisely, an unital sequence  $1, a_1, a_2, \dots$  is associated to the sequence  $1, \alpha, \alpha^2, \dots$  of powers of  $\alpha$  through a linear functional  $E$  that looks like the expectation of a r.v.. So the classical umbral calculus is no more than a symbolic tool by which handling number sequences. An umbra looks as the framework of a r.v. with no reference to any probability space, somehow getting closer to statistical methods. Compared with previous symbolic methods employed in statistics, see for example [1] and [20], by a theoretical point of view it has the advantage to reduce the combinatorics of symmetric functions, commonly used by statisticians, to few relations which cover a great variety of calculations [11]. By a computational point of view, the efficiency of umbral calculus in manipulating expressions involving r.v.'s has been tested on the theory of  $k$ -statistics [10] and their generalizations [12] as well as in manipulating  $U$ -statistics and product moments of sample moments [11]. Recently, also the free cumulant theory has been approached by means of this new syntax [13], showing promises for future developments [14].

Finally, the employment of the classical umbral calculus in corrections of moments for grouped data has one more advantage. Except for the papers of Craig [5] and Baten [3], no attention was paid to multivariate generalizations of Sheppard's corrections, probably due to the complexity of the resulting formulae. The notion of multiset is the combinatorial device that, via the symbolic method, gives rise to a closed-form formula for multivariate parent distributions that could be implemented in few steps by using any symbolic package.

The paper is structured as follows. Section 2 is provided for readers un-

aware of the classical umbral calculus. Let us underline that the theory of the classical umbral calculus has now reached a more advanced level compared to the elements here resumed. We have chosen to recall terminology, notation and the basic definitions strictly necessary to deal with the object of this paper. In particular, we recall the notion of the Bernoulli umbra, as introduced in [22]. The Bernoulli umbra is the keystone for the umbral handling of moment corrections of grouped data. In Section 3, Sheppard's corrections are given and extended to discrete parent distributions. For the continuous case, the key is to represent integrals by means of suitable umbral Bernoulli polynomials. For the discrete case, the key is the generalization of the so-called *multiplication theorem* to umbral Bernoulli polynomials. Differently from the literature on this subject, where first the expressions of corrected moments in terms of raw ones are deduced and then these expressions are inverted by solving a linear system, here we deduce directly the corrections on the moments, due to closed-form formulae. Section 4 is devoted to the multivariate generalizations of Sheppard's corrections. Some concluding remarks end the paper.

## 2 Background on umbral calculus

In the following, terminology, notation and some basic definitions of the classical umbral calculus are recalled, as introduced by Rota and Taylor in [22] and subsequently developed by Di Nardo and Senato in [8] and [9]. We skip any proof: the reader interested in-depth analysis is referred to the papers quoted below.

The classical umbral calculus is a syntax consisting of the following data:

- i) a set  $A = \{\alpha, \beta, \dots\}$ , called the *alphabet*, whose elements are named *umbrae*;
- ii) the polynomial ring  $\mathcal{R} = \mathbb{R}[x]$  in the indeterminate  $x$  with  $\mathbb{R}$  the field of real numbers<sup>2</sup>;
- iii) a linear functional  $E : \mathcal{R}[A] \rightarrow \mathcal{R}$ , called *evaluation*, such that  $E[1] = 1$  and  $E[x^n \alpha^i \beta^j \dots \gamma^k] = x^n E[\alpha^i] E[\beta^j] \dots E[\gamma^k]$  for any set of distinct umbrae in  $A$  and for  $n, m, i, j, \dots, k$  nonnegative integers (the so-called *uncorrelation property*);

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<sup>2</sup>For the aim of this paper, we need something more of the usual commutative integral domain whose quotient field is of characteristic zero, required in [22]. Due to the framework we deal, it is more convenient to refer directly to the field of real numbers  $\mathbb{R}$ .

- iv) an element  $\varepsilon \in A$ , called *augmentation*, such that  $E[\varepsilon^n] = \delta_{0,n}$ , for any nonnegative integer  $n$ , where  $\delta_{i,j} = 1$  if  $i = j$ , otherwise being zero;
- v) an element  $u \in A$ , called *unity umbra*, such that  $E[u^n] = 1$ , for any nonnegative integer  $n$ .

A sequence  $a_0 = 1, a_1, a_2, \dots$  in  $\mathcal{R}$  is umbrally represented by an umbra  $\alpha$  when

$$E[\alpha^i] = a_i, \quad \text{for } i = 0, 1, 2, \dots$$

The elements  $a_i$  are called *moments* of the umbra  $\alpha$ , in analogy with the moments of a r.v.  $X$ . In particular, an umbra is said to be *scalar* if the moments are elements of  $\mathbb{R}$  while it is said to be *polynomial* if the moments are polynomials of  $\mathbb{R}[x]$ .

An umbral polynomial is a polynomial  $p \in \mathcal{R}[A]$ . The support of  $p$  is the set of all umbrae occurring in  $p$ . If  $p$  and  $q$  are two umbral polynomials then  $p$  and  $q$  are *uncorrelated* if and only if their supports are disjoint. We said that  $p$  and  $q$  are *umbrally equivalent* if and only if

$$E[p] = E[q], \quad \text{in symbols } p \simeq q.$$

It is possible that two distinct umbrae represent the same sequence of moments, in such case they are called *similar umbrae*. More formally two umbrae  $\alpha$  and  $\gamma$  are *similar* when  $\alpha^n$  is umbrally equivalent to  $\gamma^n$ , for all  $n = 0, 1, 2, \dots$  in symbols

$$\alpha \equiv \gamma \Leftrightarrow \alpha^n \simeq \gamma^n \quad n = 0, 1, 2, \dots$$

Given a sequence  $1, a_1, a_2, \dots$  in  $\mathcal{R}$  there are infinitely many distinct, and thus similar umbrae, representing the sequence.

Two umbrae  $\alpha$  and  $\gamma$  are said to be *inverse* to each other when  $\alpha + \gamma \equiv \varepsilon$ . We denote the inverse of the umbra  $\alpha$  by  $-1.\alpha$ . Note that they are uncorrelated. Recall that, in dealing with a saturated<sup>3</sup> umbral calculus, the inverse of an umbra is not unique, but any two umbrae inverse to any given umbra are similar.

**The Bernoulli umbra.** A definition of the Bernoulli umbra  $\iota$  is given in [22]: up to similarity, the Bernoulli umbra is the unique umbra such that

$$(\iota + 1)^{n+1} \simeq \iota^{n+1} \tag{2}$$

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<sup>3</sup>Roughly speaking, a saturated umbral calculus is defined when the alphabet  $A$  is extended with a set including all auxiliary umbrae like  $-1.\alpha$ . In [22], a formal definition of saturated umbral calculus is given.

for all positive integers  $n$ . Then, the Bernoulli umbra  $\iota$  turns to be the unique (up to similarity) umbra such that  $E[\iota^n] = B_n$ , for  $n = 0, 1, 2, \dots$ , where  $\{B_n\}$  are the Bernoulli numbers. By using equivalence (2), the main properties of the Bernoulli numbers can be easily proved, as for example  $B_{2n+1} = 0$  for all nonnegative integers  $n$ . Here, we just recall that the *Bernoulli polynomials*  $\{B_n(x)\}$  are such that

$$B_n(x) = \sum_{k=0}^n \binom{n}{k} B_k x^{n-k} \simeq (x + \iota)^n. \quad (3)$$

The Bernoulli polynomials are characterized to have an average value of 0 over the interval  $[0, 1]$  for all nonnegative integers  $n$ , that is

$$\int_0^1 B_n(x) dx = 0. \quad (4)$$

We sketch a simple “*umbral*” proof. Since by simple computations we have  $\int_0^1 B_n(x) dx = E \left[ \int_0^1 (x + \iota)^n dx \right]$ , then  $\int_0^1 B_n(x) dx \simeq \int_0^1 (x + \iota)^n dx \simeq [(\iota + 1)^{n+1} - \iota^{n+1}]/(n+1) \simeq 0$ , as  $(\iota + 1)^{n+1} - \iota^{n+1} \simeq 0$ , due to equivalence (2). The approach here introduced allows us to manage integrals by using suitable umbral polynomials.

**Theorem 2.1.** *If  $-1.\iota$  is the inverse of the Bernoulli umbra and  $\{B_n(x)\}$  are the Bernoulli polynomials, then*

$$E[B_n(-1.\iota)] = \int_0^1 B_n(x) dx, \quad (5)$$

for all nonnegative integers  $n$ .

*Proof.* Via equivalence (3), we have  $B_n(-1.\iota) \simeq (-1.\iota + \iota)^n \simeq \varepsilon^n$ .  $\square$

**Corollary 2.2.** *If  $p(x) \in \mathcal{R}$  and  $h \in \mathbb{R} \setminus \{0\}, c \in \mathbb{R}$  then*

$$E[p(-1.\iota)] = \int_0^1 p(u) du, \quad E[p(-1.(h\iota) + c)] = \frac{1}{h} \int_c^{c+h} p(t) dt. \quad (6)$$

*Proof.* Without loss of generality, assume  $p(x)$  a polynomial of degree  $n$ , for some nonnegative integer  $n$ , that is  $p(x) = \sum_{k=0}^n c_k x^k$ . By recalling that  $E[(-1.\iota)^k] = 1/(k+1)$  for all nonnegative integers  $k$  [22], we have

$$E[p(-1.\iota)] = \sum_{k=0}^n \frac{c_k}{k+1} = \sum_{k=0}^n c_k \int_0^1 x^k dx = \int_0^1 p(x) dx.$$

The latter of equations (6) follows from the former. Indeed in  $\int_c^{c+h} p(t) dt$  replace  $t$  by  $h u + c$ . The result follows by recalling the equivalence  $-1.(h\iota) \equiv h(-1.\iota)$ , proved in [8] for any umbra  $\alpha \in A$ .  $\square$

In the following, we give an umbral proof of the so-called *multiplication theorem* [21] for the umbral Bernoulli polynomials. We will use this property in the next section.

**Theorem 2.3.** *If  $m$  is a nonnegative integer, then for all nonnegative integers  $n$*

$$\left(x + \frac{\iota}{m}\right)^n \simeq \frac{1}{m} \sum_{k=0}^{m-1} \left(x + \frac{k}{m} + \iota\right)^n. \quad (7)$$

*Proof.* Since  $-1.\iota + \iota \equiv \varepsilon$ , we have

$$\left(x + \frac{\iota}{m}\right)^n \simeq \left(-1.\iota + \frac{\iota}{m} + x + \iota\right)^n \simeq \sum_{j=0}^n \binom{n}{j} (x + \iota)^{n-j} \left(-1.\iota + \frac{\iota}{m}\right)^j. \quad (8)$$

By using the former of equations (6), we have

$$\left(-1.\iota + \frac{\iota}{m}\right)^j \simeq \int_0^1 \left(x + \frac{\iota}{m}\right)^j dx \simeq \sum_{k=0}^{m-1} \int_{\frac{k}{m}}^{\frac{k+1}{m}} \left(x + \frac{\iota}{m}\right)^j dx.$$

If we set  $y = mx - k$ , then

$$\sum_{k=0}^{m-1} \int_{\frac{k}{m}}^{\frac{k+1}{m}} \left(x + \frac{\iota}{m}\right)^j dx \simeq \frac{1}{m^{j+1}} \sum_{k=0}^{m-1} \int_0^1 (y + k + \iota)^j dy \simeq \frac{1}{m} \sum_{k=0}^{m-1} \left(\frac{k}{m}\right)^j.$$

The result follows by substituting these last two equivalences in (8).  $\square$

By linearity, if  $p(x) \in \mathcal{R}$ , then  $p\left(x + \frac{\iota}{m}\right) \simeq \frac{1}{m} \sum_{k=0}^{m-1} p\left(x + \frac{k}{m} + \iota\right)$ .

### 3 Corrections to grouped moments

Usually, the  $n$ -th moment  $\tilde{a}'_n$  of the grouped distribution is represented by

$$\tilde{a}'_n = \sum_{i \in \mathbb{Z}} \xi_i^n P\left(\xi_i - \frac{h}{2} < X < \xi_i + \frac{h}{2}\right), \quad (9)$$

where  $P(\cdot)$  denotes the parent distribution,  $\xi_i$  are the mid-point of classes partitioning the range and  $h$  is the width of each class. In practice, we know an estimate of  $\tilde{a}'_n$ , since when  $\tilde{a}'_n$  is computed, the probability

$$P\left(\xi_i - \frac{h}{2} < X < \xi_i + \frac{h}{2}\right)$$

is replaced by the frequency  $N_i/N$  of the corresponding class and only a finite number of classes is considered (so the summation is over a finite number of terms). In establishing approximate relations between the set of the raw moments  $a_n$  and the set of grouped moments  $\tilde{a}'_n$ , a way to avoid any assumption other than the existence of the involved moments is to employ another set of constants  $\tilde{a}_n$ . These constants  $\tilde{a}_n$  are the average of  $\tilde{a}'_n$ , when the set  $\{\xi_i\}$  is replaced by a suitable set of random points obtained by assuming random the rounding lattice. By the umbral method, we will prove that the expression of the raw moments  $a_n$  in terms of the constants  $\tilde{a}_n$  gives the corrections to grouped moments. The discussion on the nature of approximation in replacing  $\tilde{a}_n$  by  $\tilde{a}'_n$  goes beyond the aim of this paper, and it has been already tackled in the literature, see for example [2, 17] and [29].

**Continuous parent distribution: Sheppard's corrections.** Let  $f$  be a continuous probability density function of a r.v.  $X$  over  $(-\infty, \infty)$ . As usual,

$$a_n = \int_{-\infty}^{\infty} t^n f(t) dt \quad (10)$$

denotes the  $n$ -th moment of  $X$  about the origin. In the following, we assume that all absolute moments exist. The moments calculated from the grouped frequencies are given by

$$\tilde{a}_n = \frac{1}{h} \int_{-\infty}^{\infty} t^n \int_{-\frac{1}{2}h}^{\frac{1}{2}h} f(t+x) dx dt. \quad (11)$$

Indeed in (9), suppose to replace  $\xi_i$  by  $\xi_i + U$ , with  $U$  an uniform r.v. over  $(-\frac{1}{2}h, \frac{1}{2}h)$ . Equation (11) follows by setting  $\tilde{a}_n = E[\tilde{a}'_n(U)]$ .

**Theorem 3.1** (Sheppard's correction). *If the sequence  $\{\tilde{a}_n\}$  in (11) is umbrally represented by the umbra  $\tilde{\alpha}$  and the sequence  $\{a_n\}$  in (10) is umbrally represented by the umbra  $\alpha$ , then*

$$\tilde{\alpha} \equiv \alpha + h \left( -1.\iota - \frac{1}{2} \right). \quad (12)$$



*Proof.* We have

$$E[(\alpha + x)^n] = \sum_{k=0}^n \binom{n}{k} E[\alpha^k] x^{n-k} = \int_{-\infty}^{\infty} \sum_{k=0}^n \binom{n}{k} t^k x^{n-k} f(t) dt,$$

so that  $(\alpha + x)^n \simeq \int_{-\infty}^{\infty} (t + x)^n f(t) dt$ , for all nonnegative integers  $n$ . Since

$$\tilde{a}_n = \frac{1}{h} \int_{-\infty}^{\infty} \int_{-\frac{1}{2}h}^{\frac{1}{2}h} (t + x)^n f(t) dt dx = \frac{1}{h} \int_{-\frac{1}{2}h}^{\frac{1}{2}h} E[(\alpha + x)^n] dx,$$

and due to the linearity of  $E$

$$\frac{1}{h} \int_{-\frac{1}{2}h}^{\frac{1}{2}h} E[(\alpha + x)^n] dx = E \left[ \frac{1}{h} \int_{-\frac{1}{2}h}^{\frac{1}{2}h} (\alpha + x)^n dx \right],$$

equation (12) follows by observing

$$\frac{1}{h} \int_{-\frac{1}{2}h}^{\frac{1}{2}h} (\alpha + x)^n dx \simeq \left( -1 \cdot (h\iota) + \alpha - \frac{1}{2}h \right)^n,$$

where the last equivalence follows from the latter of (6) with  $c = -h/2$ .  $\square$

As a first corollary, we get equation (1). Note that from equivalence (12), we have

$$\alpha \equiv \tilde{\alpha} + h \left( \iota + \frac{1}{2} \right). \quad (13)$$

Then, by using the binomial expansion and by applying the linear functional  $E$ , for all nonnegative integers  $n$  we have

$$a_n = \sum_{j=0}^n \binom{n}{j} E \left[ \left( \iota + \frac{1}{2} \right)^j \right] h^j \tilde{a}_{n-j}.$$

In order to recover equation (1), all we need is to prove that  $E \left[ \left( \iota + \frac{1}{2} \right)^j \right] = B_j(2^{1-j} - 1)$  for all nonnegative  $j$ . This is done in the following proposition.

**Proposition 3.2.** *If  $\iota$  is the Bernoulli umbra, then for all nonnegative  $j$*

$$\left( \iota + \frac{1}{2} \right)^j \simeq (2^{1-j} - 1) \iota^j. \quad (14)$$

*Proof.* Note that for all nonnegative integers  $j$  we have

$$\begin{aligned} \left(\iota + \frac{1}{2}\right)^j &\simeq \left(-1 \cdot \frac{\iota'}{2} + \frac{\iota'}{2} + \iota + \frac{1}{2}\right)^j \simeq \left[\iota + \frac{1}{2}(-1 \cdot \iota' + 1) + \frac{\iota'}{2}\right]^j \\ &\simeq \sum_{k=0}^j \binom{j}{k} \left(\iota + \frac{\iota'}{2}\right)^{j-k} \frac{1}{2^k} (-1 \cdot \iota' + 1)^k, \end{aligned} \quad (15)$$

where  $\iota$  and  $\iota'$  denote uncorrelated Bernoulli umbrae. For all nonnegative integers  $k$ , from equation (6) we have

$$(-1 \cdot \iota' + 1)^k \simeq \int_0^1 (x+1)^k dx = \frac{2^{k+1}}{k+1} - \frac{1}{k+1} \simeq 2^{k+1}(-1 \cdot \iota)^k - (-1 \cdot \iota')^k.$$

Substituting this last equivalence in (15), we have

$$\begin{aligned} \left(\iota + \frac{1}{2}\right)^j &\simeq \sum_{k=0}^j \binom{j}{k} \left(\iota + \frac{\iota'}{2}\right)^{j-k} \left[2(-1 \cdot \iota)^k - (-1 \cdot \frac{\iota'}{2})^k\right] \\ &\simeq 2 \sum_{k=0}^j \binom{j}{k} (-1 \cdot \iota)^k \left(\iota + \frac{\iota'}{2}\right)^{j-k} - \sum_{k=0}^j \binom{j}{k} \left(-1 \cdot \frac{\iota'}{2}\right)^k \left(\frac{\iota'}{2} + \iota\right)^{j-k} \\ &\simeq 2 \left(-1 \cdot \iota + \iota + \frac{\iota'}{2}\right)^j - \left(-1 \cdot \frac{\iota'}{2} + \frac{\iota'}{2} + \iota\right)^j \simeq 2 \left(\frac{\iota'}{2}\right)^j - \iota^j, \end{aligned}$$

by which equivalence (14) follows.  $\square$

As a second corollary of Theorem 3.1, we recover equations giving the  $n$ -th moment of grouped data in terms of raw moments. Indeed, in equivalence (12), by using the binomial expansion and by applying the linear functional  $E$ , we have

$$\tilde{a}_n = \sum_{j=0}^n \binom{n}{j} E \left[ \left(-1 \cdot \iota - \frac{1}{2}\right)^j \right] h^j a_{n-j}, \quad (16)$$

for all nonnegative integers  $n$ . As before, we need to evaluate the moments of  $-1 \cdot \iota - \frac{1}{2}$ . This can be done by using only equation (6). Indeed, for all nonnegative integers  $j$  we have

$$E \left[ \left(-1 \cdot \iota - \frac{1}{2}\right)^j \right] = \int_0^1 \left(x - \frac{1}{2}\right)^j dx = \begin{cases} 0, & \text{if } j \text{ is odd,} \\ \frac{1}{j+1} \left(\frac{1}{2}\right)^j, & \text{if } j \text{ is even.} \end{cases} \quad (17)$$

So from equation (16) and by using (17), we have

$$\tilde{a}_n = \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n}{2j} \left(\frac{h}{2}\right)^{2j} \frac{a_{n-2j}}{2j+1}. \quad (18)$$

**Remark 3.3.** Theorem 3.1 still holds, when the moments are referred to a parent distribution over  $(a, b)$ . In (10) and (11), instead of using the domain of integration  $(-\infty, \infty)$ , we integrate over  $(a, b)$  and we do the same in the proof of Theorem 3.1. In (9), we refer the summation to  $i = 1, \dots, p$ , with  $p$  the number of classes partitioning  $(a, b)$  with width  $h$ . Here the r.v.  $\tilde{a}'_n(U)$  is obtained by replacing  $\xi_i$  with  $a + (i - \frac{1}{2})h + U$ .

**Discrete parent distribution.** Assume that  $m$  equidistant consecutive values of a discrete r.v. are grouped into a frequency class of width  $h$ . The  $m$  smaller intervals of width  $h/m$  go to make up the class width  $h$  in such a way that the  $m$  values of the variable represent the mid-points of the sub-intervals. Without loss of generality, we assume that

$$P\left(X = \frac{ih}{m}\right) = \frac{h}{m}f\left(\frac{ih}{m}\right), \quad i \in \mathbb{Z},$$

with

$$f\left(\frac{ih}{m}\right) \geq 0 \quad \forall i \in \mathbb{Z} \quad \text{and} \quad \sum_{i \in \mathbb{Z}} \frac{h}{m}f\left(\frac{ih}{m}\right) = 1.$$

In (9), replace  $\xi_i$  by

$$\left(mi + U - \frac{m-1}{2}\right) \frac{h}{m},$$

with  $U$  a r.v. which has any of  $m$  possible values  $0, 1, \dots, m-1$  equally probable. The  $m$  values of  $\tilde{a}'_n(U)$  corresponding to the  $m$  distinct methods of grouping a discrete distribution are

$$\tilde{a}'_n(U) = \sum_{i \in \mathbb{Z}} \left[ \left(mi + U - \frac{m-1}{2}\right) \frac{h}{m} \right]^n \sum_{j=0}^{m-1} \frac{h}{m}f\left[\left(mi + U - j\right) \frac{h}{m}\right].$$

By doing some calculations, we recover the expression of moments calculated from the grouped frequencies

$$\tilde{a}_n = E[\tilde{a}'_n(U)] = \frac{1}{m} \sum_{j=0}^{m-1} \frac{h}{m} \sum_{s \in \mathbb{Z}} \left( \frac{sh}{m} - \frac{m-1-2j}{2m}h \right)^n f\left(\frac{sh}{m}\right). \quad (19)$$

In the following, we denote by  $a_n$  the  $n$ -th moment of the discrete parent distribution, that is

$$a_n = \frac{h}{m} \sum_{s \in \mathbb{Z}} \left( \frac{sh}{m} \right)^n f\left(\frac{s}{m}\right). \quad (20)$$

**Theorem 3.4** (Corrections). *If the sequence  $\{\tilde{a}_n\}$  in (19) is umbrally represented by the umbra  $\tilde{\alpha}$  and the sequence  $\{a_n\}$  in (20) is umbrally represented by the umbra  $\alpha$ , then*

$$\tilde{\alpha} \equiv \alpha + h \left( -1.\iota - \frac{1}{2} \right) + \frac{h}{m} \left( \iota + \frac{1}{2} \right). \quad (21)$$

*Proof.* By linearity, we have

$$E \left[ \left( \alpha - \frac{m-1-2j}{2m} h \right)^n \right] = \frac{h}{m} \sum_{s \in \mathbb{Z}} \left( \frac{sh}{m} - \frac{m-1-2j}{2m} h \right)^n f \left( \frac{sh}{m} \right).$$

The result follows by observing that

$$\begin{aligned} \tilde{\alpha}_n &\simeq \frac{1}{m} \sum_{j=0}^{m-1} \left( \alpha - \frac{m-1-2j}{2m} h \right)^n \\ &\simeq \frac{h^n}{m} \sum_{j=0}^{m-1} \left( \left\{ -1.\iota + \frac{\alpha}{h} - \frac{m-1}{2m} \right\} + \frac{j}{m} + \iota \right)^n \\ &\simeq h^n \left( -1.\iota + \frac{\alpha}{h} - \frac{m-1}{2m} + \frac{\iota}{m} \right)^n, \end{aligned} \quad (22)$$

where equivalence (22) follows from equivalence (7), by replacing  $x$  with  $\left\{ -1.\iota + \frac{\alpha}{h} - \frac{m-1}{2m} \right\}$ . Suitably rearranging the terms, we obtain equivalence (21).  $\square$

It is interesting to compare equivalence (12) with (21). In this last equivalence, we find just one addend more, which is an umbra whose moments are given in Proposition 3.2. Obviously, if  $m \rightarrow \infty$  from (21) we recover (12), as stated by Craig in [5]. Moreover, since from (21) we find

$$\alpha \equiv \tilde{\alpha} + h \left( \iota + \frac{1}{2} \right) + \frac{h}{m} \left( -1.\iota - \frac{1}{2} \right), \quad (23)$$

equivalence (23) differs from equivalence (13) for an umbra whose moments are given in (17). These observations turn to be useful when we seek expression of raw moments  $\{a_n\}$  in terms of grouped moments  $\{\tilde{a}_n\}$ . To the best of our knowledge, the last version available is given by Craig in [5], but its structure is quite complex and involves integer partitions. Here we give a different expression. By applying the binomial expansion and the linear functional  $E$  to equivalence (23), for all nonnegative integers we have

$$a_n = \sum_{k=0}^n \binom{n}{k} E \left[ \left\{ \tilde{\alpha} + h \left( \iota + \frac{1}{2} \right) \right\}^{n-k} \right] E \left[ \left( -1.\iota - \frac{1}{2} \right)^k \right] \frac{h^k}{m^k}. \quad (24)$$

Moments of  $\tilde{\alpha} + h(\iota + \frac{1}{2})$  are given by Sheppard's corrections (1) in the continuous case, while the moments of  $-1.\iota - \frac{1}{2}$  are given in (17). So all we need is to replace these expressions in equation (24):

$$a_n = \sum_{k=0}^{\lceil \frac{n}{2} \rceil} \binom{n}{2k} \left(\frac{h}{2m}\right)^{2k} \frac{1}{2k+1} \sum_{j=0}^{n-2k} \binom{n-2k}{j} (2^{1-j} - 1) B_j h^j \tilde{a}_{n-2k-j}.$$

By a **Maple** procedure, it is straightforward to verify that the first eight moments  $\{a_1, \dots, a_8\}$ , computed by means of this formula, are the same as given by Craig in [5].

## 4 Corrections to multivariate grouped data

In [10], it has been shown that the notion of multiset is the key for dealing with multivariate moments in umbral syntax. Here we recall briefly the notations.

Let  $M$  be a multiset of umbral monomials. A *multiset*  $M$  is a pair  $(\bar{M}, f)$ , where  $\bar{M}$  is a set, called the *support* of the multiset, and  $f$  is a function from  $\bar{M}$  to nonnegative integers. For each  $\mu \in \bar{M}$ ,  $f(\mu)$  is the *multiplicity* of  $\mu$ . We denote a multiset  $(\bar{M}, f)$  simply by  $M$ . When the support of  $M$  is a finite set, say  $\bar{M} = \{\mu_1, \mu_2, \dots, \mu_j\}$ , we write

$$M = \{\mu_1^{(f(\mu_1))}, \mu_2^{(f(\mu_2))}, \dots, \mu_j^{(f(\mu_j))}\} \quad \text{or} \quad M = \{\underbrace{\mu_1, \dots, \mu_1}_{f(\mu_1)}, \dots, \underbrace{\mu_j, \dots, \mu_j}_{f(\mu_j)}\}.$$

Set

$$\mu_M = \prod_{\mu \in \bar{M}} \mu^{f(\mu)}. \quad (25)$$

For example, if  $M = \{\mu_1^{(2)}, \mu_2^{(1)}, \mu_3^{(4)}\}$ , we denote by  $\mu_M$  the product  $\mu_1^2 \mu_2 \mu_3^4$ .

A *multivariate moment* is the element of  $\mathcal{R}$  corresponding to the umbral monomial  $\mu_M$  via the evaluation  $E$ , i.e.

$$E[\mu_M] = m_{t_1 \dots t_j}, \quad (26)$$

where  $t_i = f(\mu_i)$  for  $i = 1, 2, \dots, j$ . For example if  $M = \{\mu_1^{(2)}, \mu_2^{(1)}, \mu_3^{(4)}\}$ , we have  $E[\mu_M] = m_{214}$ . When the umbral monomials  $\mu_i$  are uncorrelated,  $m_{t_1 \dots t_j}$  becomes the product of the moments of  $\mu_i$ . More details on the meaning and the use of the symbol  $\mu_M$  are given in [10].

Suppose  $\mathbf{X} = (X_1, X_2, \dots, X_j)$  a multivariate r.v. with the joint density function  $f_{\mathbf{X}}(\mathbf{x})$  over  $\mathbb{R}^j$ . Note that by using the same arguments, we can deal with any range of bounded rectangle type. As usual

$$m_{t_1 \dots t_j} = \int_{\mathbb{R}^j} x_1^{t_1} \cdots x_j^{t_j} f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x} \quad (27)$$

denotes the multivariate moment of  $\mathbf{X}$  of order  $(t_1, \dots, t_j)$ . The moments calculated from the grouped frequencies are given by

$$\tilde{m}_{t_1 \dots t_j} = \frac{1}{h_1 \cdots h_j} \int_{R_j} \int_{\mathbb{R}^j} (x_1 + z_1)^{t_1} \cdots (x_j + z_j)^{t_j} f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x} d\mathbf{z}, \quad (28)$$

where

$$R_j = \left\{ \mathbf{z} = (z_1, \dots, z_j) \in \mathbb{R}^j : z_k \in \left( -\frac{1}{2}h_k, \frac{1}{2}h_k \right) \forall k \in \{1, \dots, j\} \right\}$$

and  $\{h_k\} \in \mathbb{R} \setminus \{0\}$  are the width window for any component. A proof of (28) can be done similarly to the one sketched for the univariate case (9).

**Theorem 4.1** (Multivariate Sheppard's correction). *If the sequence  $\{\tilde{m}_{t_1 \dots t_j}\}$  in (28) is umbrally represented by the umbral monomial  $\tilde{\mu}_M$ , with  $M$  a multiset of finite support  $\{\mu_1, \dots, \mu_j\}$ , and the sequence  $\{m_{t_1 \dots t_j}\}$  in (27) is umbrally represented by the umbral monomial  $\mu_M$ , then*

$$\tilde{\mu}_M \equiv \left[ \mu + h \left( -1.\iota - \frac{1}{2} \right) \right]_M \quad (29)$$

where we set

$$\left[ \mu + h \left( -1.\iota - \frac{1}{2} \right) \right]_M = \prod_{k=1}^j \left[ \mu_k + h_k \left( -1.\iota_k - \frac{1}{2} \right) \right]^{t_k}$$

with  $\{\iota_k\}$  uncorrelated Bernoulli umbrae.

*Proof.* Due to linearity, we have

$$E[(\mu_1 + z_1)^{t_1} \cdots (\mu_j + z_j)^{t_j}] = \int_{\mathbb{R}^j} (x_1 + z_1)^{t_1} \cdots (x_j + z_j)^{t_j} f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x}.$$

Moreover, from (28) we have

$$\begin{aligned} \tilde{m}_{t_1 \dots t_j} &\simeq \frac{1}{h_1 \cdots h_j} \int_{R_j} (\mu_1 + z_1)^{t_1} \cdots (\mu_j + z_j)^{t_j} d\mathbf{z}, \\ &\simeq \prod_{k=1}^j \frac{1}{h_k} \int_{-\frac{1}{2}h_k}^{\frac{1}{2}h_k} (\mu_k + z_k)^{t_k} dz_k \simeq \prod_{k=1}^j \left( \mu_k + h_k \left( -1.\iota_k - \frac{1}{2} \right) \right)^{t_k} \end{aligned}$$

by which the result follows.  $\square$

In the support of the multiset  $M$ , if we choose

$$\mu_k = \tilde{\mu}_k + h_k \left( \iota_k + \frac{1}{2} \right), \quad k = 1, \dots, j$$

then equivalence (29) becomes an identity and so by notation (26)

$$\mu_M \equiv \left[ \tilde{\mu} + h \left( \iota + \frac{1}{2} \right) \right]_M \quad (30)$$

where again

$$\left[ \tilde{\mu} + h \left( \iota + \frac{1}{2} \right) \right]_M = \prod_{k=1}^j \left[ \tilde{\mu}_k + h_k \left( \iota_k + \frac{1}{2} \right) \right]^{t_k}.$$

Equivalence (30) can be easily implemented in any symbolic package. Indeed, in order to recover expressions of raw multivariate moments in terms of grouped moments, we need to multiply summations like

$$\sum_{s_k=1}^n \binom{n}{s_k} \tilde{\mu}_k^{s_k} h_k^{n-s_k} (2^{1-n+s_k} - 1) B_{n-s_k}$$

corresponding to the  $n$ -th power of  $\tilde{\mu}_k + h_k \left( \iota_k + \frac{1}{2} \right)$ , and then replace occurrences of products like  $\tilde{\mu}_1^{s_1} \tilde{\mu}_2^{s_2} \dots \tilde{\mu}_j^{s_j}$  with  $\tilde{m}_{s_1 \dots s_j}$ .

For the sake of brevity, we skip the details of the proof of corrections when the multivariate parent distribution is discrete. This can be done taking the same way used for the univariate parent distribution, as we have done for the continuous case. Here, the corrections to moments due to the grouping can be formulated in umbral terms as:

$$\mu_M \equiv \left[ \tilde{\mu} + h \left( \iota + \frac{1}{2} \right) + \frac{h}{m} \left( -1 \cdot \iota - \frac{1}{2} \right) \right]_M, \quad (31)$$

where, by the symbol in the right hand side of the previous equivalence, we denote the following product

$$\prod_{k=1}^j \left[ \tilde{\mu}_k + h_k \left( \iota_k + \frac{1}{2} \right) + \frac{h_k}{m_k} \left( -1 \cdot \iota_k - \frac{1}{2} \right) \right],$$

where  $m_k$  are the number of consecutive values grouped in a frequency class of width  $h_k$ . The multivariate version of similarity (21) is

$$\tilde{\mu}_M \equiv \left[ \mu + \frac{h}{m} \left( \iota + \frac{1}{2} \right) + h \left( -1 \cdot \iota - \frac{1}{2} \right) \right]_M. \quad (32)$$

**Concluding remarks.** The main goal of this paper is to show how the corrections of moments resulting from grouping into classes may be summarized in few closed-form formulae. Moreover, the multivariate formulae can be constructed from the univariate ones, by a suitable indexing of umbral monomials with a multiset.

Once more, this paper shows how the classical umbral calculus should be taken into account for managing sequence of numbers related to r.v.'s, since many calculations are reduced. For example, the reader interested in recovering corrections for cumulants and factorial moments, by using the classical umbral calculus, can refer to [9]. The noteworthy simplification in the expression of corrections, when referred to cumulants instead of moments, was first pointed out by Langdon and Ore [19] for a continuous parent distribution over  $(-\infty, \infty)$ . By using the umbral syntax introduced for the  $\alpha$ -cumulant umbra and the  $\alpha$ -factorial umbra, it is possible to recover these corrections by one line proof both for continuous and for discrete parent distributions.

The umbral techniques applied to Sheppard's corrections open the way to deal with new problems that would be interesting to explore. Indeed, the umbral version of Sheppard's corrections, here introduced, refers to the averaging interpretation of these corrections, as proposed by Craig [5]. Different interpretations give rise to different forms of Sheppard's corrections, see for example [24]. It would be interesting to see if the umbral method simplifies the calculation apparatus and add some new formulae also for these different interpretations.

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